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THE MINIMUM DIMENSIONS OF THE CONTROL VECTOR IN THE LINEAR DYNAMIC PROBLEM OF STABILIZATION*

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A mathematical formalization is proposed of the problem of estimating the number of engines necessary to stabilize a mechanical elastic system, functioning in conditions of zero gravity, in a specified position. Conditions are given which allow the class of control matrices imparting the property of full controllability to dynamic systems to be described /1/. The analysis of conditions of full controllability for mechanical systems in the neighbourhood of the position of equilibrium was given in /2/.

We consider the following dynamic selfsimilar system:

$$x^{\cdot} = Fx + Gu,\ x \in R^n,\ u \in R^m \tag{1}$$

where F and G are constant matrices, x is the state vector and u the control vector. We know /1, 2/ that if the condition of full controllability

$$\operatorname{rank} \| G,\ FG,\ F^2G, \ldots, F^{n-1}G \| = n \tag{2}$$

holds, then a control $u(t)$ exists which takes the system (1) from any initial position x_0 to the origin of coordinates. If condition (2) does not hold, then such a control does not, in general, exist. Our aim is to determine the minimum number of scalar control functions u_i, i.e. the minimum dimensions of the control vector for which the condition of full controllability can be attained by a suitable choice of the control matrix G. The answer to this problem is given by the following theorem.

Theorem. Let k_i be the number of linearly independent eigenvectors corresponding to the i-th eigenvalue of the matrix F, and $k = \max_i k_i$. Then k will be the minimum dimension of the control vector $u(t)$ for which the choice of the matrix G can still result in satisfying the condition of complete controllability (2).

Following /3/, we shall introduce a number of concepts and assertions. We shall call the vector g the root vector corresponding to the eigenvalue λ_i, provided that

$$(F - \lambda_i E)^h g = 0 \tag{3}$$

for some integral value of $h > 0$. We shall call the heigt j of the vector g the smallest value of h for which condition (3) holds, i.e. $(F - \lambda_i E)^{j-1} g \neq 0$ and $(F - \lambda_i E)^j g = 0$. The zero vector has zero height by definition. The set of root vectors corresponding to some eigenvalue λ_i, forms a root subspace P_i, invariant under the transformation $F - \lambda_i E$, and consequently also invariant under the operator F. The root subspace P_i in turn decomposes into k_i cyclic subspaces (k_i is the number of linearly independent eigenvectors corresponding to the i-th eigenvalue), invariant under the operator F. These subspaces Π_μ^i, $\mu = 1, 2, \ldots k_i$, are stretched over the vectors $g_{\mu\nu}^i$, which satisfy the condition

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$$g^i_{\mu\,\nu+1} = (F - \lambda_i E)\, g_{\mu\nu}{}^i,\ \nu = 1, 2, \ldots, q_\mu - 1,\ (F - \lambda_i E)\, g^i_{\mu q_\mu} = 0$$

where q_μ is the height of the vector generating $\Pi_\mu{}^i$. The set of vectors $g_{\mu\nu}{}^i, \nu = 1, 2, \ldots, q_\mu$ forms the μ-th tower in the subspace P_i, is linearly independent, and q_μ is the height of the μ-th tower. The maximum height q of the tower in P_i is equal to the multiplicity λ_i, representing the root of the minimum polynomial cancelling the matrix F, and the sum of the tower heights is equal to the multiplicity λ_i representing the root of the characteristic polynomial. The vectors $g_{\mu\nu}{}^i$, $\mu = 1, 2, \ldots, k_i$, $\nu = 1, 2, \ldots, q_\mu$ are linearly independent and form a canonical basis in P_i. The operator F in this basis has a canonical Jordan form.

The following theorem is well-known /3/: let Q_r be a cyclic subspace for the operator F, generated by the vector G^r. Then Q_r will be a direct sum of the cyclic subspaces generated by the projections of the vector G^r on the root subspaces, and the dimensions of the cyclic subspace will be equal to the sum of the heights of the projections of the generating vector on the root subspaces.

Therefore the condition of complete controllability (2) can be formulated as follows. Let G^r be the r-th column of the control matrix $G(1)$. In order for the dynamic system (1) to be completely controllable it is necessary and sufficient that the geometrical sum of the cyclic subspaces generated by the projections of the vectors $G^r, r = 1, 2, \ldots, m$, on every root subspace coincide with this subspace.

We shall now prove the theorem on the minimum dimensions of the control vector. According to the theorem in /3/ quoted above, it is sufficient to consider the cyclic subspaces generated by the projections of the vectors G^r, $r = 1, \ldots, m$, on an arbitrary root subspace P_i.

We shall denote the projection of the vector G^r on the root subspace P_i by $G_i{}^r$ and explain the conditions under which the geometrical sum of the cyclic subspaces generated by the vectors $G_i{}^r$, $r = 1, \ldots, m$, can coincide with P_i.

We shall show that if the dimension of the control vector $m \geqslant k = \max_i k_i$, then we can always select a matrix G ensuring that the condition of complete controllability holds. We shall take, in the root subspace in question, a set of k_i vectors $g_{r1}{}^i, r = 1, 2, \ldots, k_i$, where every vector $g_{r1}{}^i$ generates a corresponding tower, and write $G_i{}^r = g_{r1}{}^i$, $r = 1, 2, \ldots, k_i$, $G_i{}^r = 0$, $r > k_i$. Then the geometrical sum of the cyclic subspaces generated by the vectors $G_i{}^r, r = 1, 2, \ldots, m$ will coincide, by virtue of their choice, with the root subspace P_i. From the relation $(F - \lambda_i E)^h g_{r1} = 0$, $r = 1, 2, \ldots, k_i$ it follows that $(F - \bar{\lambda}_i E)^h \bar{g}_{r1} = 0$. Therefore, if the vector g_{r1} generates the r-th tower in a root subspace, then a complex conjugate vector $\bar{g}_{r1}$ will also generate a tower in some root subspace. Therefore, the columns $G^r, r = 1, 2, \ldots, m$ representing the control matrices G obtained as a result of summing the corresponding projections $G_i{}^r$ over all root subspaces P_i, will be real as required.

Now we shall show that if condition $m < k_i = k$, holds for some root subspace P_i, then the geometrical sum of the cyclic subspaces generated by the vectors $G_i{}^r, r = 1, 2, \ldots, m$, will not coincide with the root subspace for any choice of G.

To arrive at the proof, we shall consider, in the root subspace P_i, the sequence of columns $\mathbf{g}_s$, $s = 1, 2, \ldots, q$, where q is the maximum height of the towers appearing in the root subspace in question. The column $\mathbf{g}_1 = \mathrm{col}\,(g_{11}, g_{21}, \ldots, g_{k1})$ is composed of linearly independent vectors $g_{\mu 1}$, $\mu = 1, 2, \ldots, k$ where every vector $g_{\mu 1}$ generates a corresponding tower in the cyclic root space, and the set of vectors occurring in the tower comprises the basis in P_i. We shall assume that the symbol $(F - \lambda_i E)\,\mathbf{g}_s$ denotes a column of vectors $(F - \lambda_i E)\,\mathbf{g}_s = \mathrm{col}\,((F - \lambda_i E)\, g_{1s}, (F - \lambda_i E)\, g_{2s}, \ldots, (F - \lambda_i E)\, g_{ks})$. Let us define the sequence of columns $\mathbf{g}_s$ by the relation

$$\mathbf{g}_{s+1} = (F - \lambda_i E)\,\mathbf{g}_s,\ s = 1, 2, \ldots, q-1,\ \mathbf{g}_{q+1} = (F - \lambda_i E)\,\mathbf{g}_q = 0 \tag{4}$$

Hence, we obtain the set of vectors $g_{\mu s}$, $\mu = 1, 2, \ldots, k$. The vectors form a system of vectors, some of which may be null vectors in the root space. Let us denote by $\mathbf{G}_0$ the column of m vectors where the r-th component of $\mathbf{G}_0$ is a projection $G_i{}^r$, $r = 1, 2, \ldots, m$ of the vector G^r on the subspace P_i. The column of vectors $\mathbf{G}_0 = \mathrm{col}\,(G_i{}^1, G_i{}^2, \ldots, G_i{}^m)$ can be expanded in a sum in the components of the columns $\mathbf{g}_s$ by virtue of the completeness of the vectors $g_{\mu s}$ in the space P_i:

$$\mathbf{G}_0 = \sum_{s=1}^{g} A_s{}^i \mathbf{g}_s \tag{5}$$

where $A_s{}^i - (m \times k)$ are matrices. Let us apply the operator $F - \lambda_i E$ to every column comprising the right and left side of the relation (5). By virtue of relation (4) and the definition of the operation $(F - \lambda_i E)\,\mathbf{g}_s$, we obtain

$$(F - \lambda_i E)\,\mathbf{G}_0 = \sum_{s=1}^{g-1} A_s{}^i \mathbf{g}_{s+1} \tag{6}$$

$$(F - \lambda_i E)^2 \mathbf{G}_0 = \sum_{s=1}^{g-2} A_s{}^i \mathbf{g}_{s+2},\ \ldots,\ (F - \lambda_i E)^{g-1} \mathbf{G}_0 = A_1{}^i \mathbf{g}_g$$

We shall show that when $m < k$, the space stretched over the vectors constituting the columns

$$G_0,\ (F - \lambda_i E)\,G_0, \ldots, (F - \lambda_i E)^{n-1} G_0 \qquad (7)$$

does not coincide with the root space. Let us denote by L the subspace stretched over the vectors constituting the column g_1. Its dimensions will be equal to k.

Then the dimensions of the subspace L_1, stretched over the vectors $A_1{}^i g_1$, will not exceed $m < k, L_1 \subset L$. Let us choose in L a vector ξ, orthogonal to L_1. Then it will be clear that we shall not be able to represent the vector ξ in the form of a linear combination of vectors from the sequence (7), since by definition it is not a linear combination of vectors occurring in the column G_0, and all remaining terms of the sequence will contain, according to (6), only the vectors not belonging to L. The projection on P_i of the space stretched over the column vectors of the matrix (2) coincides with the space stretched over the set of vectors (7), and the latter space is smaller than the root space. Therefore, the condition of complete controllability (2) cannot be satisfied.

From what was said above, it follows that the necessary and sufficient condition of complete controllability of system (1) is that condition

$$\operatorname{rank} A_1{}^i = k_i \qquad (8)$$

holds for all root subspaces P_i, where $A_1{}^i$ is given by formula (5) and k_i is the number of linearly independent eigenvectors corresponding to the eigenvalue λ_i.

As we showed before, the inequality $\operatorname{rank} A_1{}^i < k_i$ leads to violation of the condition of complete controllability. Let condition (8) hold for all root subspaces. We shall show that the space stretched over the set of vectors (7) coincides with the root space P_i. We denote the subspace stretched over the set of vectors $(F - \lambda_i E)^j G_0,\ (F - \lambda_i E)^{j+1} G_0, \ldots, (F - \lambda_i E)^{q-1} G_0$ (6) by V_j, and the subspace stretched over the set of vectors $g_{j+1}, g_{j+2}, \ldots, g_q$ (4) by W_j. Now using reverse induction in j, we shall show that W_j and V_j coincide when $0 \leqslant j \leqslant q-1$, provided that condition (8) holds. When $j = q-1$, the coincidence of the subspaces is obvious by virtue of the property of the matrix $A_1{}^i$ (8) and formula (6). Let us assume that the subspaces V_j and W_j coincide when $q-1 > j \geqslant 1$, and prove that V_{j-1} and W_{j-1} coincide. We know /3/ that W_{j-1} is a straight sum of the subspace W_j and a subspace stretched over the vectors of the column g_j, and hence stretched over the vectors of the column $A_1{}^i g_j$. According to formula (6) we have

$$(F - \lambda_i E)^{j-1} G_0 = A_1{}^i g_j + \xi_j$$

where ξ_j is a column of vectors belonging to W_j. From this, together with the inductive assumption, we conclude that the space V_{j-1} is a straight sum of the subspace stretched over the vectors of the column $A_1{}^i g_j$, and the subspace W_j. Therefore V_{j-1} also coincides with W_{j-1}. Finally, we have that V_0 coincides with W_0. In other words, the goemetrical sum of the cyclic subspaces generated by the projections of the vectors G^r, $r = 1, 2, \ldots, m$, on the root subspace coincides with this subspace, and this proves that the condition of complete controllability (2) holds.

Example. We shall consider a mechanical system consisting of two identical pendulums suspended from identical dollies, and an engine whose type is immaterial. The dollies can move along a horizontal guide. The moment at the engine shaft is transmitted to the dolly wheels and directly to the pendulums. What was proved above shows that it is impossible to design a transmission system which will ensure the damping of arbitrary oscillations of both pendulums by the action of an engine. Here it is important that the control matrix should be independent of time, otherwise a single engine could be used to quench the oscillations first of one pendulum, and then of the other.

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